CHAPTER 5 – RELATIONSHIPS WITHIN TRIANGLES

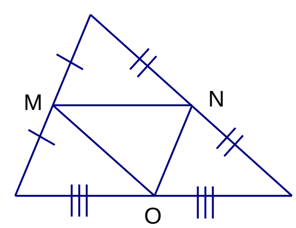
In this chapter we address three **Big IDEAS**:

1) ***Using properties of special segments in triangles***
2) ***Using triangle inequalities to determine what triangles are possible***
3) ***Extending methods for justifying and proving relationships***

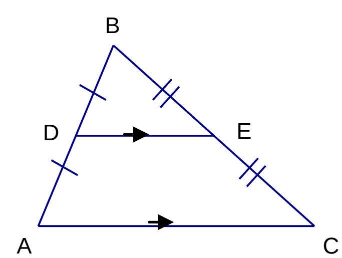
Section:	**5 – 1 Midsegment Theorem**
Essential Question	**What is a midsegment of a triangle?**

Warm Up:

Key Vocab:

Midsegment of a Triangle	A segment that connects the midpoints of two sides of the triangle. **Example:** $\overline{MO}, \overline{MN}, \overline{NO}$ are *midsegments*	

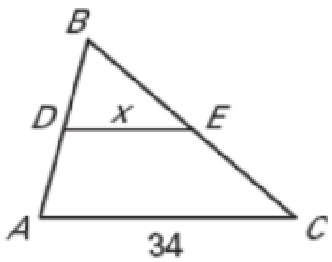
Theorem:

Midsegment Theorem	
The segment connecting the midpoints of two sides of a triangle is a. parallel to the third side b. and is half as long as that side.	 $\overline{DE} \parallel \overline{AC}$ and $DE = \frac{1}{2}AC$

Show:

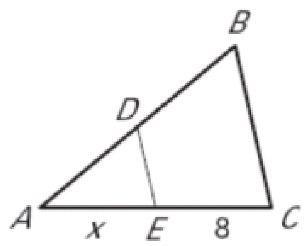
Ex 1: If $\overline{DE}$ is the midsegment of $\triangle ABC$, find the value of *x*.

a.



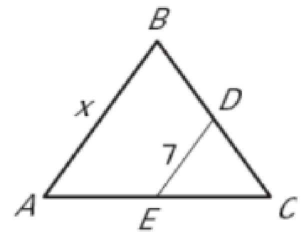
$x = \frac{1}{2} \cdot 34 = 17$

b.



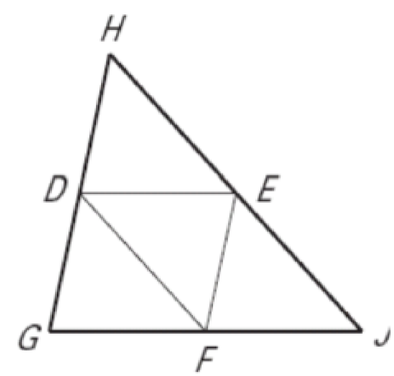
$x = 8$

c.



$x = 2 \cdot 7 = 14$

Ex 2: In $\triangle GHJ$, *D*, *E*, and *F* are midpoints of the sides.



a. If *DE* = 8 and *GJ* = 3*x*, find *GJ*.

$$2DE = GJ$$
$$16 = 3x$$
$$5.\overline{3} = x$$

$$GJ = 3(5.\overline{3}) = 16$$

b. If *EF* = 2*x* and *GH* = 12, find *EF*.

$$2EF = GH$$
$$2(2x) = 12$$
$$x = 3$$

$$EF = 2(3) = 3$$

c. If *HJ* = 8*x* − 2 and *DF* = 2*x* + 11, find *HE*.

$$2DF = HJ$$
$$2(2x+11) = 8x-2$$
$$4x+22 = 8x-2$$
$$24 = 4x$$
$$6 = x$$

$$HE = DF = 2(6)+11 = 23$$

d. If *HD* = 3*x* + 29 and *DG* = 14*x* + 7, find *EF*.

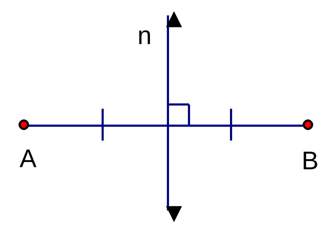
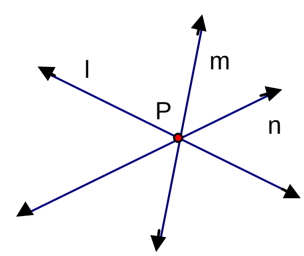
$$HD = DG$$
$$3x+29 = 14x+7$$
$$22 = 11x$$
$$2 = x$$

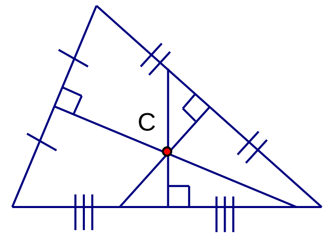
$$EF = HD = DG = 14(2)+7 = 35$$

Section:	**5 – 2 Use Perpendicular Bisectors**
Essential Question	How do you find the point of concurrency of the perpendicular bisectors of the sides of triangle?

Warm Up:

Key Vocab:

Perpendicular Bisector	A segment, ray, line, or plane, that is perpendicular to a segment at its midpoint.	 Line n is $\perp$ *bisector* of $\overline{AB}$
Concurrent	Three or more lines, rays, or segments that intersect in the same point	 Lines l, m, and n are concurrent
Point of Concurrency	The intersection point of concurrent lines, rays, or segments.	Point p is the point of concurrency

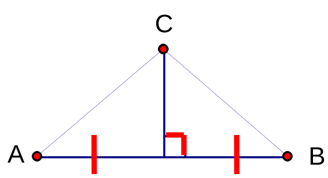
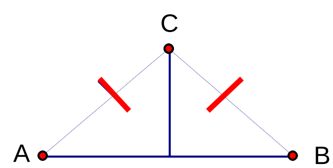
<table>
<tr>
<td>Circumcenter</td>
<td>The point of concurrency of the three perpendicular bisectors of the triangle</td>
<td>

Point C is the circumcenter</td>
</tr>
</table>

Theorems:

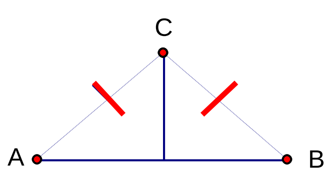
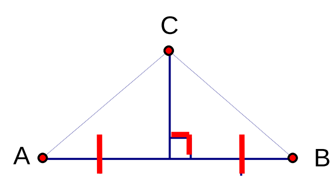
<table>
<tr><th colspan="2">Perpendicular Bisector Theorem</th></tr>
<tr>
<td>In a plane, if

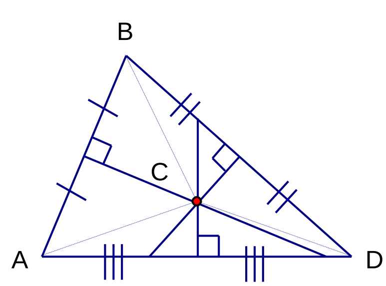
a point is on the perpendicular bisector of a segment,</td>
<td>then

it is equidistant from the endpoints of the segment.</td>
</tr>
<tr>
<td>Point C is on the $\perp$ bisector of $\overline{AB}$</td>
<td>$\overline{AC} \cong \overline{BC}$</td>
</tr>
<tr>
<td>
</td>
<td>
</td>
</tr>
</table>

<table>
<tr><th colspan="2">Converse of the Perpendicular Bisector Theorem</th></tr>
<tr>
<td>In a plane, if

a point is equidistant from the endpoints of a segment,</td>
<td>then

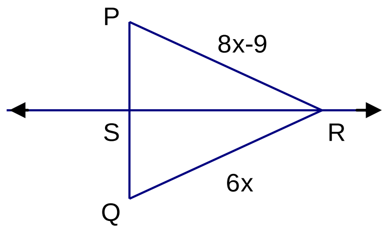
it is on the perpendicular bisector of the segment.</td>
</tr>
<tr>
<td>$\overline{AC} \cong \overline{BC}$</td>
<td>Point C is on the $\perp$ bisector of $\overline{AB}$</td>
</tr>
<tr>
<td>
</td>
<td>
</td>
</tr>
</table>

<table>
<tr><th colspan="2">Circumcenter Theorem</th></tr>
<tr>
<td>The perpendicular bisectors of a triangle intersect at a point that is equidistant from the vertices of the triangles</td>
<td>

$\overline{AC} \cong \overline{BC} \cong \overline{DC}$</td>
</tr>
</table>

Show:

Ex 1: In the diagram, $\overleftrightarrow{RS}$ is the *perpendicular bisector* of $\overline{PQ}$. Find *PR*.



$$8x-9=6x$$
$$2x=9 \qquad PR=8(4.5)-9=27$$
$$x=4.5$$

Ex 2: R is the circumcenter of ΔOPQ where $OS=10,\ QR=12,\ PQ=22$.

a) Find OP

20

b) Find RP

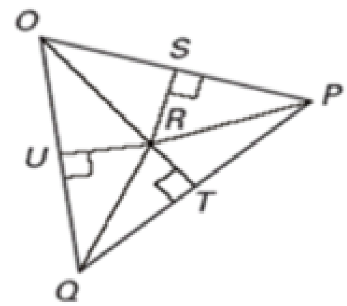
12

c) Find OR

12

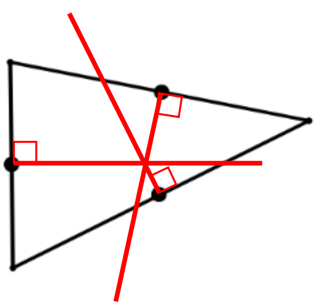
d) Find TP

11

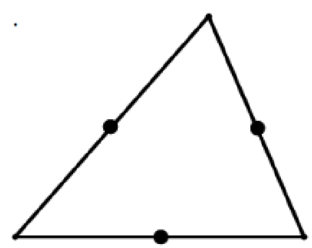


Ex 3: Draw all three perpendicular bisectors for the given triangles. The midpoint of each side has already been found. (Do not force them to go through vertices!)

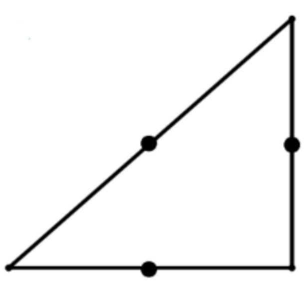
a)



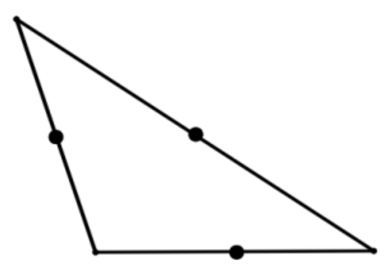
b)



c)



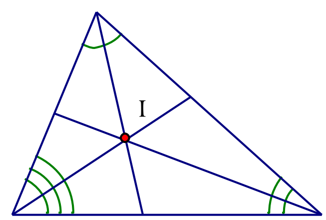
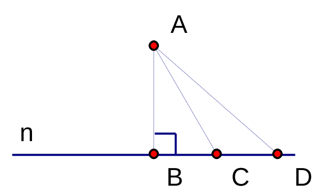
d)



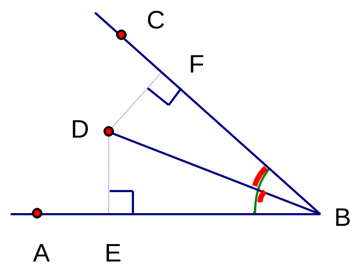
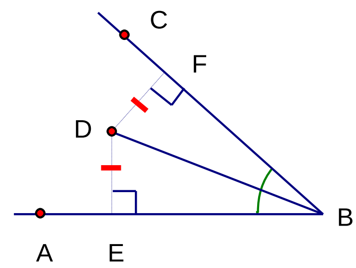
Section:	**5 – 3 Use Angle Bisectors of Triangles**
Essential Question	When can you conclude that a point is on the bisector of an angle?

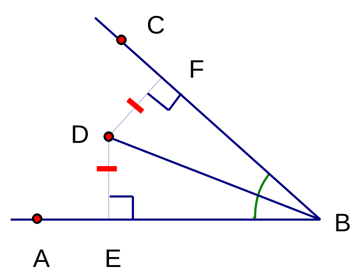
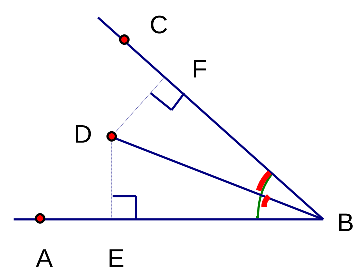
Warm Up:

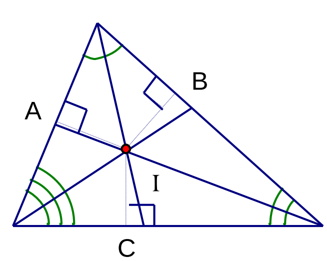
Key Vocab:

Incenter	The point of concurrency of the three angle bisectors of the triangle.	 Point *I* is the *incenter*.
Distance from a point to a line	The length of the perpendicular segment from the point to the line	 *AB* is the distance between point *A* and line *n*

Theorem:

Angle Bisector Theorem	
If a point is on the bisector of an angle,	**then** it is equidistant from the two sides of the angle.
$\overrightarrow{BD}$ bisects $\angle ABC$	$\overline{DE} \cong \overline{DF}$
	

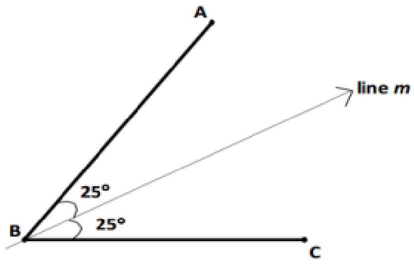
Converse of the Angle Bisector Theorem	
If a point is in the interior of an angle and is equidistant from the sides of the angle,	**then** it lies on the bisector of the angle.
$\overline{DE} \cong \overline{DF}$	$\overrightarrow{BD}$ bisects $\angle ABC$
	

Incenter Theorem	
The angle bisectors of a triangle intersect at a point that is equidistant from the ***sides*** of the triangle.	 $\overline{AI} \cong \overline{BI} \cong \overline{CI}$

Show:

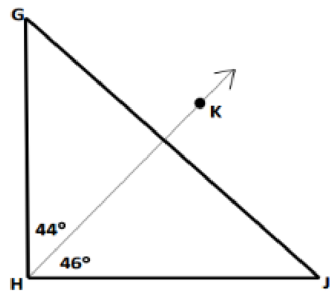
Ex 1: Which of the following ARE angle bisectors? How can you tell?

a.



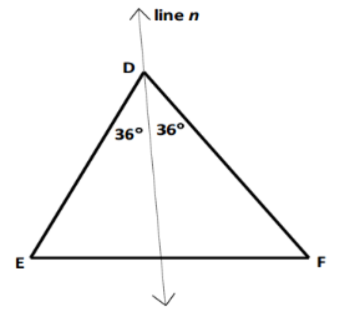
m is the angle bisector of $\angle ABC$ because it creates two congruent angles

b.



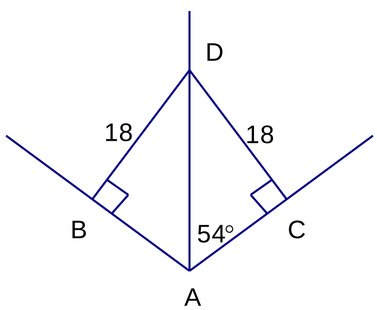
k is not the angle bisector because it does not create congruent angles.

c.



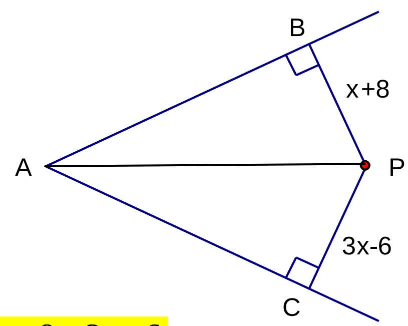
n is the angle bisector of $\angle EDF$ because it creates two congruent angles.

Ex 2: Find the measure of $\angle BAD$.



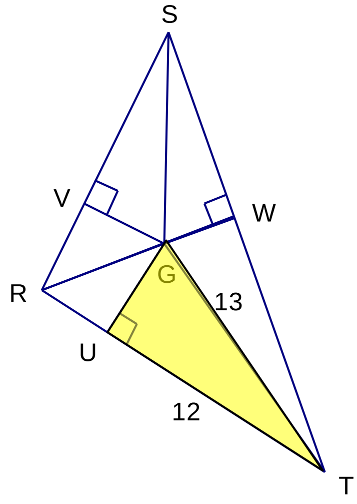
54°

Ex 3: For what value of x does P lie on the bisector of $\angle A$?



$$x+8=3x-6$$
$$14=2x$$
$$7=x$$

Ex 4: In the diagram, G is the incenter of ΔRST. Find GW.



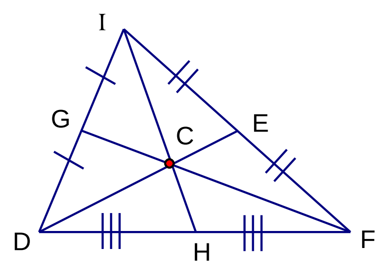
$$a^2 + b^2 = c^2$$
$$GU^2 + 12^2 = 13^2$$
$$GU = \sqrt{169 - 144}$$
$$GU = 5$$

$$GU = GW = 5$$

Section:	**5 – 4 Use Medians and Altitude**
Essential Question	How do you find the centroid of a triangle?

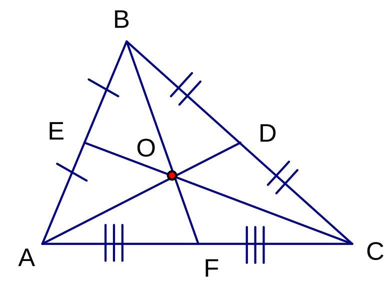
Warm Up:

Key Vocab:

<table>
<tr>
<td>Median of a Triangle</td>
<td>A segment from one vertex of the triangle to the midpoint of the opposite side</td>
<td rowspan="2">

$\overline{DE}, \overline{FG},$ and $\overline{HI}$ are medians. Point C is the centroid.</td>
</tr>
<tr>
<td>Centroid</td>
<td>The point of concurrency of the three medians of the triangle.</td>
</tr>
</table>

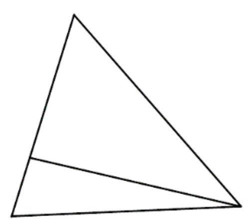
Theorems:

Centroid Theorem	
The distance between the centroid and the vertex is 2/3 the length of the median $AO = \frac{2}{3}AD; BO = \frac{2}{3}BF; CO = \frac{2}{3}CE$ The distance between the centroid and the vertex is twice the distance between the centroid and the side. $AO = 2OD; BO = 2OF; CO = 2OE$ The median is three times the length of the distance between the centroid and the side. $AD = 3OD; BF = 3OF; CE = 3OE$	

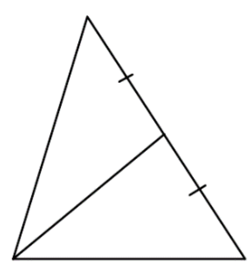
Show:

Ex 1: Which of the following are medians? How do you know?

a. No

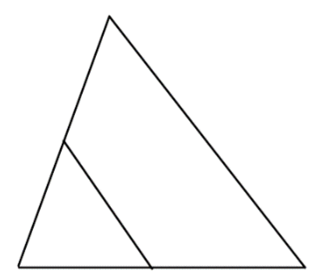


b. yes

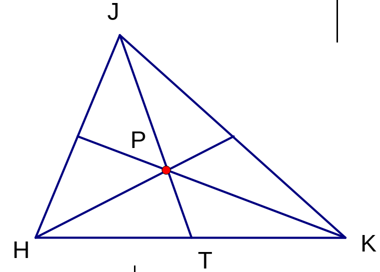


because it connects a vertex to a midpoint

c. no



Ex 2: In ΔHJK, P is the centroid



a. If $JP = 12$, find PT and JT.

$2PT = JP$

$2PT = 12$

$PT = 6$

$JT = JP + PT$

$JT = 12 + 6 = 18$

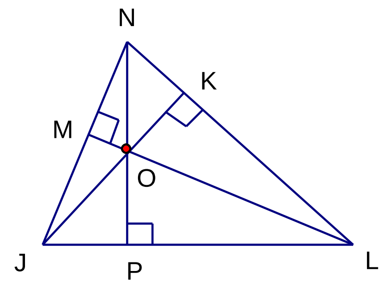
b. If $JP = x - 6$ and $JT = 2x + 2$, then $x = ?$.

$JP = \frac{2}{3}JT$

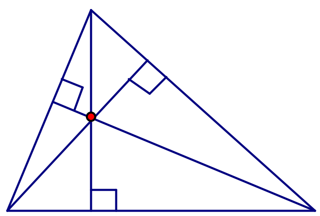
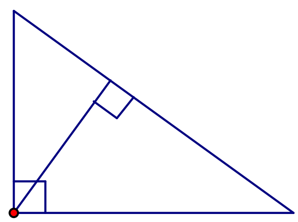
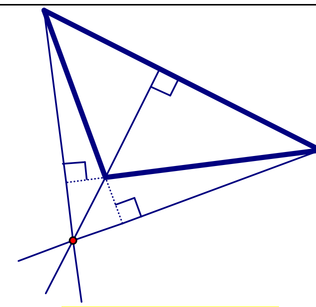
$x - 6 = \frac{2}{3}(2x + 2)$

$3x - 18 = 4x + 4$

$x = 22$

<table>
<tr><td>Altitude of Triangle</td><td>The perpendicular segment from one vertex of the triangle to the line that contains the opposite side.</td><td>

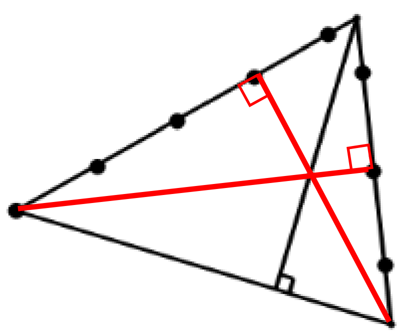
$\overline{JK}, \overline{LM},$ and $\overline{NP}$ are altitudes. Point O is the orthocenter.</td></tr>
</table>

An orthocenter can be inside, on, or outside the triangle.		
 Acute = inside	 Right = on	 Obtuse = outside

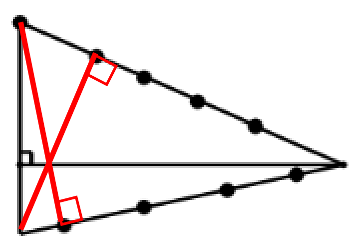
Show:

Ex 3: One of the altitudes for the given triangle has already been drawn. Draw the other two altitudes of the triangle. (The dots on the sides are given as clues to the possible endpoint)

a.

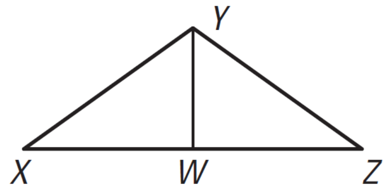


b.



Ex 4: Decide whether $\overline{YW}$ is a *perpendicular bisector, angle bisector, medians,* and/or *altitude.* Name **ALL** terms that apply.

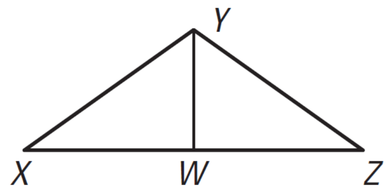
a. $\overline{YW} \perp \overline{XZ}$



altitude

---

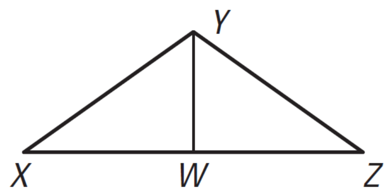
b. $\angle XYW \cong \angle ZYW$



Angle bisector

---

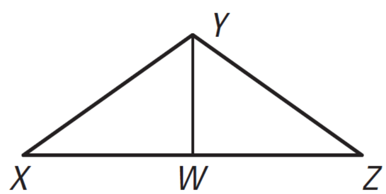
c. $\overline{XW} \cong \overline{ZW}$



median

---

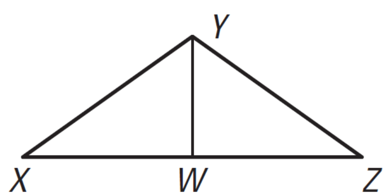
d. $\overline{YW} \perp \overline{XZ}$ and $\overline{XW} \cong \overline{ZW}$



Altitude, median, angle bisector,
perpendicular bisector

---

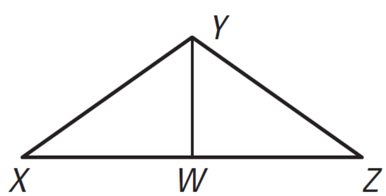
e. $\Delta XYW \cong \Delta ZYW$



Altitude, median, angle bisector,
Perpendicular bisector

---

f. $\overline{YW} \perp \overline{XZ}$ and $\overline{XY} \cong \overline{ZY}$



Altitude median, angle bisector,
Perpendicular bisector

Closure:

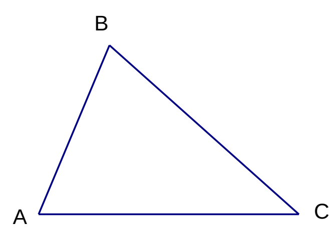
- Name the four points of concurrency of a triangle and describe how each is formed.

Section:	**5 – 5 Use Inequalities in a Triangle**
Essential Question	How do you find the possible lengths of the third side of a triangle if you know the lengths of two sides?

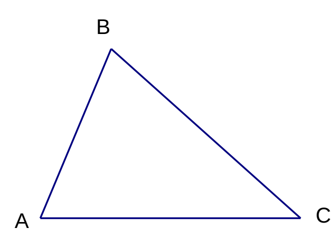
Warm Up:

Draw an obtuse scalene triangle. Find and label the largest angle and the longest side. Find and label the smallest angle and shortest side. What do you notice?

Theorems:

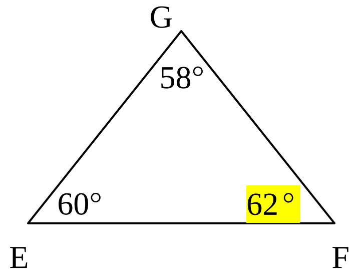


- In any triangle, the largest angle is across from the longest side.
- In any triangle, the smallest angle is across from the shortest side.

Triangle Inequality Theorem	
The sum of the lengths of any two sides of a triangle is greater than the length of the third side. If two sides of a triangle are x and y, then the third side must be between $x + y$ and $x - y$, where $x \geq y$.	

Show:

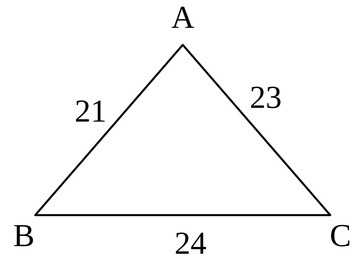
Ex 1: Identify the longest side.



$\overline{GE}$

Ex 2: Identify the smallest angle.

$\angle C$



Ex 3: A triangle has one side of length 11 and another of length 6. Describe the possible lengths of the third side.

$5 < x < 17$

Ex 4: A triangle has one side of length 11 and another of length 15. Describe the possible lengths of the third side.

$4 < x < 26$

Ex 5: Is it possible to construct a triangle with the given side lengths? If not, *explain* why not.

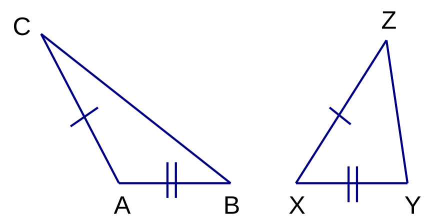
1. 6, 10, 15 yes

2. 11, 16, 32 no, $11+16=27 \ngtr 32$

Section:	**5 – 6 Inequalities in Two Triangles**
Essential Question	How do you use the SSS Inequality and SAS Inequality Theorems?

Warm Up:

Theorems:

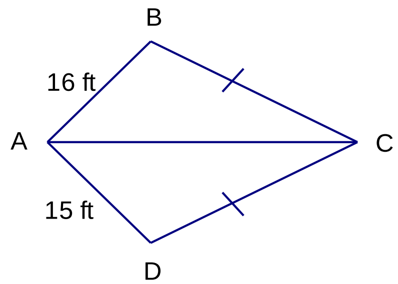


Hinge Theorem (SAS Inequality Theorem)	
If two sides of one triangle are congruent to two sides of another triangle, and the included angle of the first is larger than the included angle of the second,	**then** the third side of the first is longer than the third side of the second.
$\overline{AC} \cong \overline{XZ}, \overline{AB} \cong \overline{XY}$, AND $m\angle A > m\angle X$	$BC > YZ$

Converse of the Hinge Theorem (SSS Inequality Theorem)	
If two sides of one triangle are congruent to two sides of another triangle, and the third side of the first is longer than the third side of the second	**then** the included angle of the first is larger than the included angle of the second.
$\overline{AC} \cong \overline{XZ}, \overline{AB} \cong \overline{XY}$, AND $BC > YZ$	$m\angle A > m\angle X$

Show:

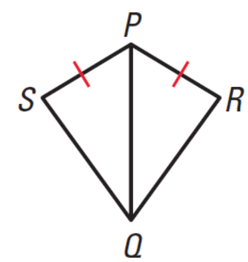
Ex 1: Given that $\overline{BC} \cong \overline{DC}$, how does $\angle ACB$ compare to $\angle ACD$?



$\angle ACB > \angle ACD$ by the SSS Inequality (Hinge Converse) Theorem

Ex 2:

a. If $PR = PS$ and $m\angle QPR > m\angle QPS$, which is longer: $\overline{SQ}$ or $\overline{RQ}$?

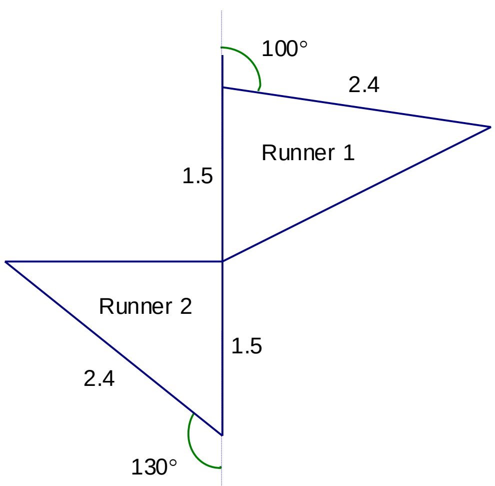


$\overline{RQ}$ by the SAS Inequality (Hinge) Theorem

b. If $PR = PS$ and $RQ < SQ$, which is larger: $\angle RPQ$ or $\angle SPQ$?

$\angle SPQ$ by the SSS Inequality (Hinge Converse) Theorem

Ex 3: Two runners start together and run in opposite directions. Each one goes 1.5 miles, changes direction, and goes 2.4 miles. The first runner starts due north and runs. 100° towards the east. The other runner starts due south and turns 130° towards the west. Both runners return to the starting point. Which runner ran farther? *Explain.*



Each triangle has side lengths 1.5 mi and 2.4 mi, and the angles between those sides are 80° and 50°. The *Hinge Theorem*, the third side of the triangle for Runner 1 is longer, so Runner 1 ran further.

Closure:

- Describe the difference between the Hinge Theorem and its Converse.