# *CHAPTER 5* – RELATIONSHIPS WITHIN TRIANGLES

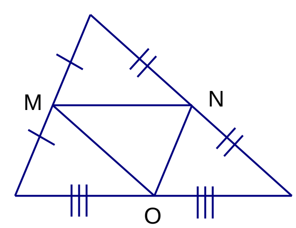
*In this chapter we address three* **Big IDEAS**:

1) ***Using properties of special segments in triangles***
2) ***Using triangle inequalities to determine what triangles are possible***
3) ***Extending methods for justifying and proving relationships***

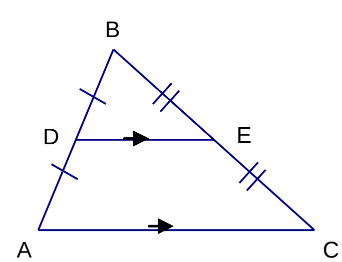
| | |
|---|---|
| Section: | **5 – 1 Midsegment Theorem** |
| Essential Question | **What is a midsegment of a triangle?** |

**Warm Up:**

**Key Vocab:**

| | | |
|---|---|---|
| **Midsegment of a Triangle** | A segment that connects the midpoints of two sides of the triangle.<br><br>**Example:** $\overline{MO}, \overline{MN}, \overline{NO}$ are *midsegments* |  |

**Theorem:**

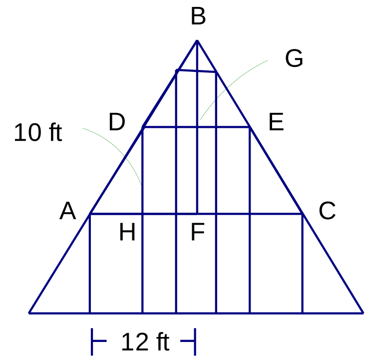
| **Midsegment Theorem** | |
|---|---|
| The segment connecting the midpoints of two sides of a triangle is<br>a. parallel to the third side<br>b. and is half as long as that side. | <br>$\overline{DE} \parallel \overline{AC}$ and $DE = \frac{1}{2}AC$ |

**Show:**

**Ex 1:** In the diagram of an A-frame house, $\overline{DG}$ and $\overline{DH}$ are midsegments of $\Delta ABC$. Find $DG$ and $BF$.

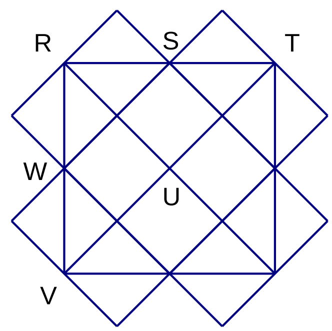
$DG = 6$ ft

$BF = 20$ ft



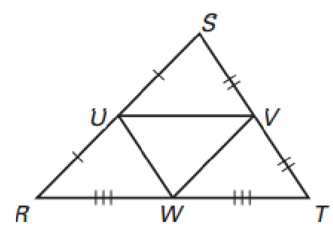

**Ex 2:** In the diagram, $\overline{RS} \cong \overline{TS}$ and $\overline{RW} \cong \overline{VW}$. Explain why $\overline{VT} \parallel \overline{WS}$

$\overline{RS} \cong \overline{TS}$ and $\overline{RW} \cong \overline{VW}$, so $S$ and $W$ are the midpoints of $\overline{RT}$ and $\overline{RV}$, and $\overline{SW}$ is a midsegment of $\Delta RTV$. Therefore, $\overline{VT} \parallel \overline{WS}$ by the *Midsegment Theorem*.




**Try it:**

**Ex 3:** Use $\Delta RST$ to answer each of the following




**a.** If If $UV = 13$, find $RT$.

$RT = 2 \bullet 13 = 26$

**b.** If the perimeter of $\Delta RST = 68$ inches, find the perimeter of $\Delta UVW$.

$\text{Perimeter}_{\Delta UVW} = \frac{1}{2} \bullet 68 = 34 \text{ inches}$

**c.** If $VW = 2x - 4$ and $RS = 3x - 3$, what is $VW$?

$2(2x-4) = 3x - 3$

$4x - 8 = 3x - 3$

$x = 5$

$VW = 2(5) - 4 = 6$

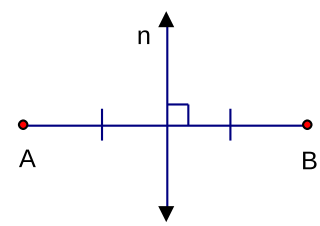
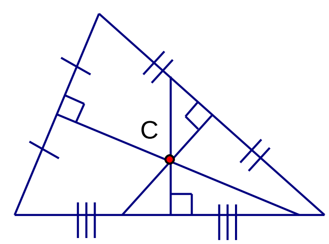
| | |
|---|---|
| Section: | **5 – 2 Use Perpendicular Bisectors** |
| Essential Question | How do you find the point of concurrency of the perpendicular bisectors of the sides of triangle? |

**Warm Up:**

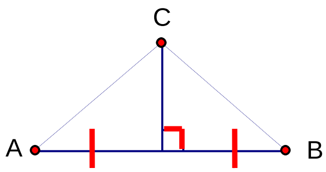
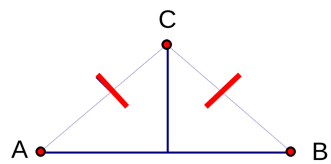
***Review Definitions for Euler Line Project:**

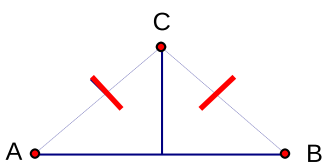
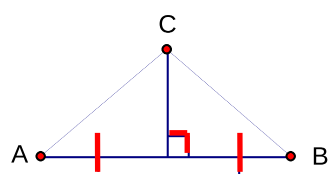
- **points of concurrency**
- **circumcenter**
- **centroid**
- **orthocenter**

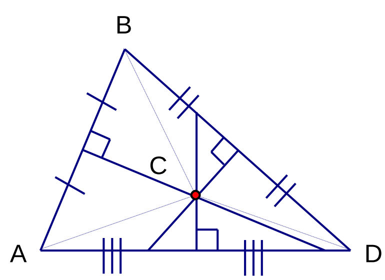
**Key Vocab:**

| | | |
|---|---|---|
| **Perpendicular Bisector** | A segment, ray, line, or plane, that is perpendicular to a segment at its midpoint. |  Line *n* is $\perp$ *bisector* of $\overline{AB}$ |
| **Circumcenter** | The point of concurrency of the three perpendicular bisectors of the triangle |  Point *C* is the *circumcenter* |

**Theorems:**

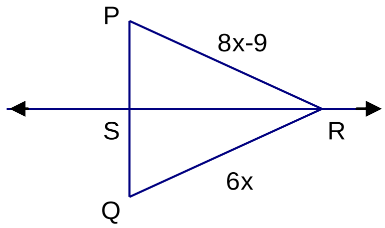
| Perpendicular Bisector Theorem | |
|---|---|
| **In a plane, if**<br><br>a point is on the perpendicular bisector of a segment, | **then**<br><br>it is equidistant from the endpoints of the segment. |
| Point $C$ is on the $\perp$ bisector of $\overline{AB}$ | $\overline{AC} \cong \overline{BC}$ |
|  |  |

| Converse of the Perpendicular Bisector Theorem | |
|---|---|
| **In a plane, if**<br><br>a point is equidistant from the endpoints of a segment, | **then**<br><br>it is on the perpendicular bisector of the segment. |
| $\overline{AC} \cong \overline{BC}$ | Point $C$ is on the $\perp$ bisector of $\overline{AB}$ |
|  |  |

| Circumcenter Theorem | |
|---|---|
| The perpendicular bisectors of a triangle intersect at a point that is equidistant from the ***vertices*** of the triangles | <br>$\overline{AC} \cong \overline{BC} \cong \overline{DC}$ |

**Show:**

**Ex 1:** In the diagram, $\overleftrightarrow{RS}$ is the perpendicular bisector of $\overline{PQ}$. Find *PR*.




$8x - 9 = 6x$

$2x = 9 \quad PR = 8(4.5) - 9 = 27$

$x = 4.5$

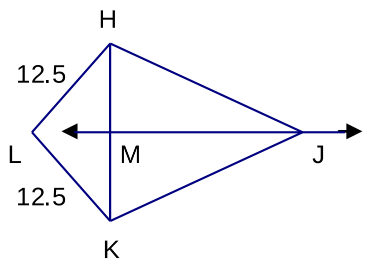
**Ex 2:** In the diagram, $\overleftrightarrow{JM}$ is the perpendicular bisector of $\overline{HK}$.

a. Which lengths in the diagram are equal?

$HM = KM; HJ = KJ; HL = KL$

b. Is *L* on $\overleftrightarrow{JM}$?

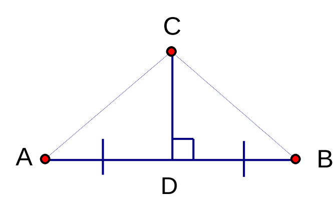
Yes




**Ex 3:** Prove the Perpendicular Bisector Theorem

Given: *C* is on the perp. bis. of $\overline{AB}$

Prove: $\overline{AC} \cong \overline{BC}$

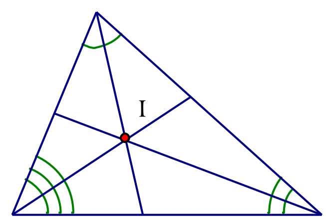



| Statements | Reasons |
|---|---|
| 1. *C* is on the perp. bis. of $\overline{AB}$ | 1. Given |
| 2. $\overline{AD} \cong \overline{DB}$ | 2. Def of segment bisector |
| 3. $\overline{CD} \cong \overline{CD}$ | 3. Reflexive Prop. |
| 4. $\angle ADC$ and $\angle BDC$ are right $\angle$'s | 4. Def of perp. |
| 5. $\angle ADC \cong \angle BDC$ | 5. Rt. Angles cong. Thm |
| 6. $\Delta ADC \cong \Delta BDC$ | 6. SAS $\cong$ Post |
| 7. $\overline{AC} \cong \overline{BC}$ | 7. CPCTC |

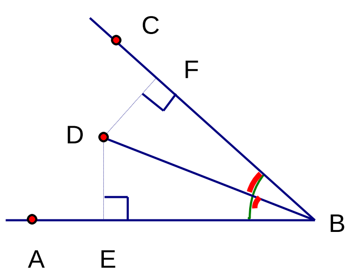
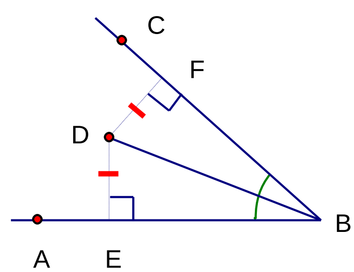
| | |
|---|---|
| Section: | **5 – 3 Use Angle Bisectors of Triangles** |
| Essential Question | When can you conclude that a point is on the bisector of an angle? |

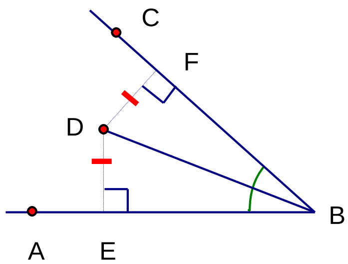
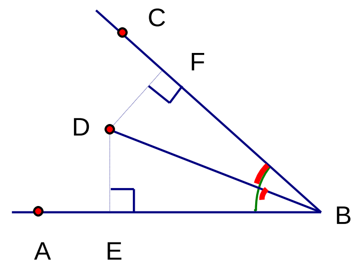
**Warm Up:**

**Key Vocab:**

| | | |
|---|---|---|
| **Incenter** | The point of concurrency of the three angle bisectors of the triangle. |  Point *I* is the *incenter*. |

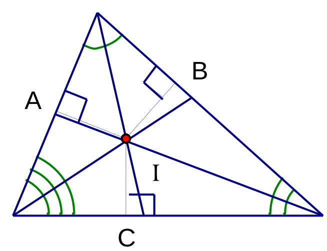
**Theorem:**

| Angle Bisector Theorem | |
|---|---|
| **If**<br>a point is on the bisector of an angle, | **then**<br>it is equidistant from the two sides of the angle. |
| $\overrightarrow{BD}$ bisects $\angle ABC$ | $\overline{DE} \cong \overline{DF}$ |
|  |  |

| Converse of the Angle Bisector Theorem | |
|---|---|
| **If**<br><br>a point is in the interior of an angle and is equidistant from the sides of the angle, | **then**<br><br>it lies on the bisector of the angle. |
| $\overline{DE} \cong \overline{DF}$ | $\overrightarrow{BD}$ bisects $\angle ABC$ |
|  |  |



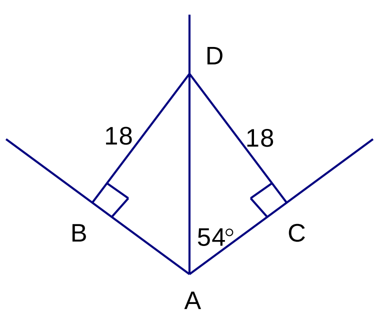


| Incenter Theorem | |
|---|---|
| The angle bisectors of a triangle intersect at a point that is equidistant from the ***sides*** of the triangle. | <br>$\overline{AI} \cong \overline{BI} \cong \overline{CI}$ |



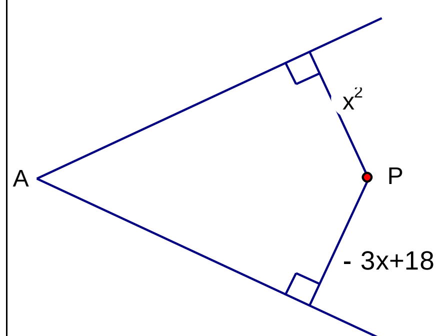
**Show:**

**Ex 1:** Find the measure of $\angle BAD$.



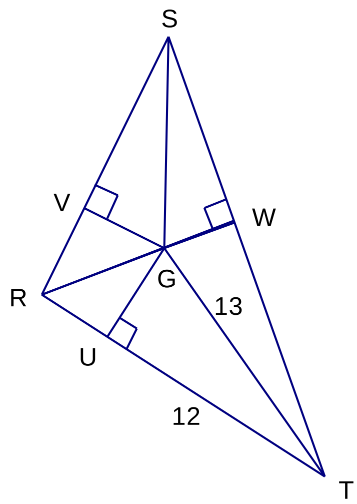

$54^{\circ}$

**Ex 2:** For what value of $x$ does $P$ lie on the bisector of $\angle A$?




$$x^2 = -3x + 18$$
$$x^2 + 3x - 18 = 0$$
$$(x+6)(x-3) = 0$$
$$x = -6, x = 3$$

**Ex 3:** In the diagram, $G$ is the incenter of $\Delta RST$. Find $GW$.




$12^2 + GU^2 = 13^2$

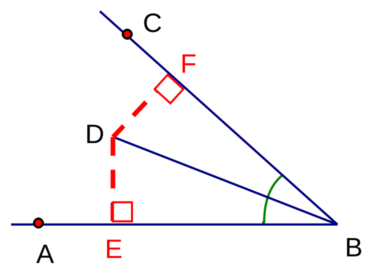
$GU = \sqrt{169 - 144}$

$GU = 5$

$GU = GV = GW$

$\therefore GW = 5$

**Ex 4:** Prove the Angle Bisector Theorem




Given: $D$ is on the angle bis. of $\angle ABC$

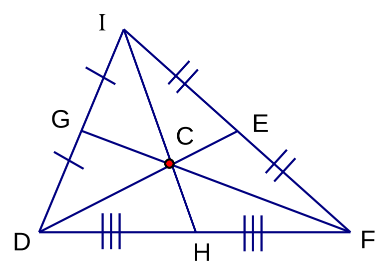
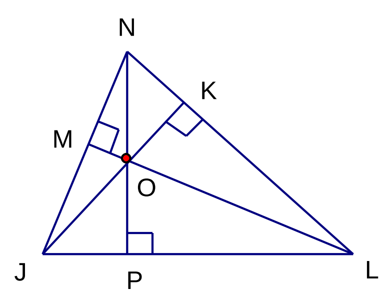
Prove: $\overline{DE} \cong \overline{DF}$

| Statements | Reasons |
|---|---|
| 1. $D$ is on the angle bis. of $\angle ABC$ | 1. Given |
| 2. Draw $\overline{DE} \perp \overline{AB}$ and $\overline{DF} \perp \overline{BC}$ | 2. Perp. Post. |
| 3. $\angle DEB$ and $\angle DFB$ are right $\angle$'s | 3. Def of Perp. |
| 4. $\angle DEB \cong \angle DFB$ | 4. Right Angles Cong. Thm. |
| 5. $\overline{BD} \cong \overline{BD}$ | 5. Reflexive Prop. |
| 6. $\angle EBD \cong \angle FBD$ | 6. Def of ang. bisector |
| 7. $\Delta EBD \cong \Delta FBD$ | 7. AAS $\cong$ Post |
| 8. $\overline{DE} \cong \overline{DF}$ | 8. CPCTC |

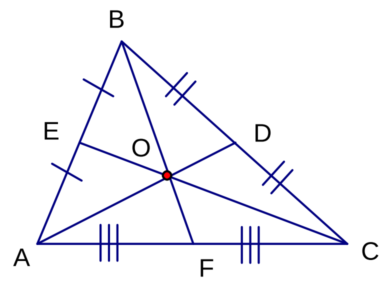
| | |
|---|---|
| Section: | **5 – 4 Use Medians and Altitude** |
| Essential Question | How do you find the centroid of a triangle? |

**Warm Up:**

**Key Vocab:**

| | | |
|---|---|---|
| **Median of a Triangle** | A segment from one vertex of the triangle to the midpoint of the opposite side | <br>$\overline{DE}, \overline{FG},$ and $\overline{HI}$ are *medians*. Point $C$ is the *centroid*. |
| **Centroid** | The point of concurrency of the three medians of the triangle. | |
| **Altitude of Triangle** | The perpendicular segment from one vertex of the triangle to the line that contains the opposite side. | <br>$\overline{JK}, \overline{LM},$ and $\overline{NP}$ are *altitudes*. Point $O$ is the *orthocenter*. |
| **Orthocenter** | The point of concurrency of the three altitudes of a triangle. | |

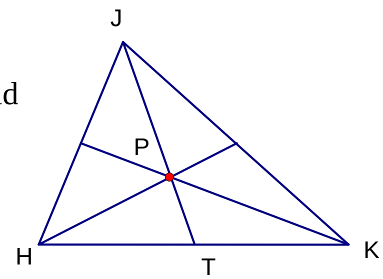
**Theorems:**

| Centroid Theorem | |
|---|---|
| The medians of a triangle intersect at a point that is two thirds of the distance from each vertex to the midpoint of the opposite side.<br><br>$AO = \frac{2}{3}AD; BO = \frac{2}{3}BF; CO = \frac{2}{3}CE$ |  |



**Show:**

**Ex 1:** In $\Delta HJK$, $P$ is the centroid




**a.** If $JP = 12$, find $PT$ and $JT$.

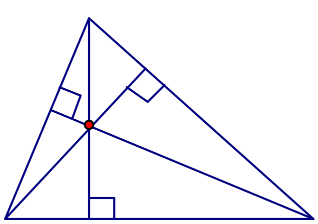
$$JP = \frac{2}{3}(JT)$$
$$12 = \frac{2}{3}(JT)$$
$$18 = JT$$

$$PT = JT - JP$$
$$PT = 18 - 12 = 6$$

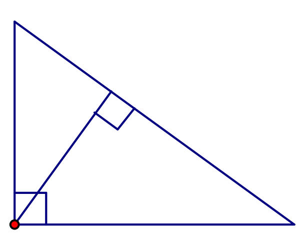
**b.** If $JP = x^2$ and $PT = 2x$, then $x = ?$.

$$x^2 = 2 \cdot 2x$$
$$x^2 = 4x$$
$$x^2 - 4x = 0$$
$$x(x-4) = 0$$
~~$x = 0$~~, $x = 4$

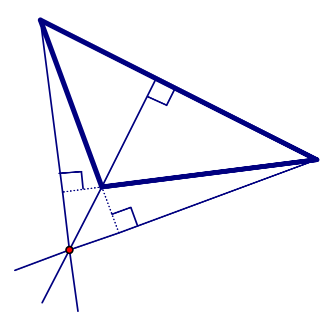
**Ex 2:** Show that the orthocenter can be inside, on, or outside the triangle.



Inside = Acute Triangle

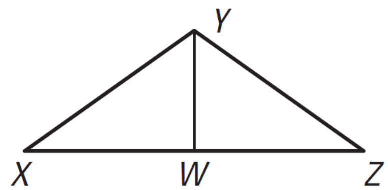


On = Right Triangle



Outside = Obtuse triangle

**Ex 3:** Decide whether $\overline{YW}$ is a *perpendicular bisector, angle bisector, medians,* and/or *altitude*. Name **ALL** terms that apply.




**a.** $\overline{YW} \perp \overline{XZ}$ Altitude

**b.** $\angle XYW \cong \angle ZYW$ angle bisector

**c.** $\overline{XW} \cong \overline{ZW}$ median

**d.** $\overline{YW} \perp \overline{XZ}$ and $\overline{XW} \cong \overline{ZW}$ perpendicular bisector, angle bisector, medians, & altitude

**e.** $\Delta XYW \cong \Delta ZYW$ perpendicular bisector, angle bisector, medians, & altitude

**f.** $\overline{YW} \perp \overline{XZ}$ and $\overline{XY} \cong \overline{ZY}$ perpendicular bisector, angle bisector, medians, & altitude

**Closure:**

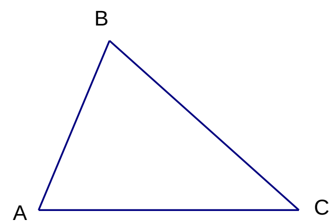
- Name the four points of concurrency of a triangle and describe how each is formed.

1. Circumcenter-intersection of the perpendicular bisectors
2. Incenter-intersection of the angle bisectors
3. Centroid-intersection of the medians
4. Orthocenter-intersection of the altitudes

| Section: | **5 – 5 Use Inequalities in a Triangle** |
|---|---|
| Essential Question | How do you find the possible lengths of the third side of a triangle if you know the lengths of two sides? |

**Warm Up:**

**Theorems:**




| **If** | **then** |
|---|---|
| one side of a triangle is longer than another side, | the angle opposite the longer side is larger than the angle opposite the shorter side. |
| $AC > AB$ | $m\angle B > m\angle C$ |
| **If** | **then** |
| one angle of a triangle is larger than another angle, | the side opposite the larger angle is longer than the side opposite the smaller angle. |
| $m\angle B > m\angle A$ | $AC > BC$ |

| Triangle Inequality Theorem | |
|---|---|
| The sum of the lengths of any two sides of a triangle is greater than the length of the third side. | $AB + BC > AC$<br>$BC + AC > AB$<br>$AC + AB > BC$ |

**Ex 1:** Three wooden beams will be nailed together to form a brace for a wall. The bottom edge of the brace is about 8 feet. One of the angles measures about $86^\circ$ and the other measure about $35^\circ$. What is the angle measure opposite the largest side of the brace?

A. $35^\circ$

B. $86^\circ$

C. $59^\circ$

D. $96^\circ$

**Ex 2:** A triangle has one side of length 11 and another of length 6. Describe the possible lengths of the third side.

$11-6=5$

$11+6=17$

Greater than 5 and less than 17

**Ex 3:** A triangle has one side of length 11 and another of length 15. Describe the possible lengths of the third side.

$15-11=4$

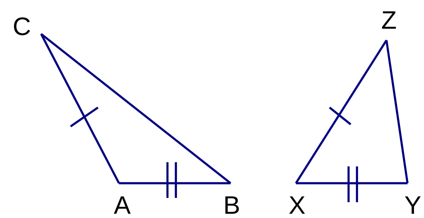
$15+11=26$

Greater than 4 and less than 26

| Section: | **5 – 6 Inequalities in Two Triangles and Indirect Proof** |
|---|---|
| Essential Question | How do you write an indirect proof? |

**Warm Up:**

**Theorems:**




**Hinge Theorem (SAS Inequality Theorem)**

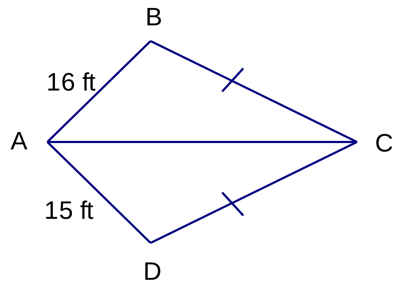
| **If** | **then** |
|---|---|
| two sides of one triangle are congruent to two sides of another triangle, and the included angle of the first is larger than the included angle of the second, | the third side of the first is longer than the third side of the second. |
| $\overline{AC} \cong \overline{XZ}, \overline{AB} \cong \overline{XY}$, AND $m\angle A > m\angle X$ | $BC > YZ$ |

**Converse of the Hinge Theorem (SSS Inequality Theorem)**

| **If** | **then** |
|---|---|
| two sides of one triangle are congruent to two sides of another triangle, and the third side of the first is longer than the third side of the second | the included angle of the first is larger than the included angle of the second. |
| $\overline{AC} \cong \overline{XZ}, \overline{AB} \cong \overline{XY}$, AND $BC > YZ$ | $m\angle A > m\angle X$ |

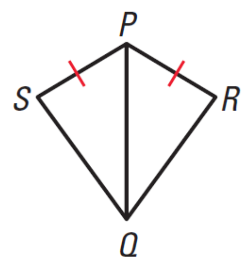
**Show:**

**Ex 1:** Given that $\overline{BC} \cong \overline{DC}$, how does $\angle ACB$ compare to $\angle ACD$?




$\angle ACB > \angle ACD$ by the SSS Inequality (Hinge Converse) Theorem

**Ex 2:**



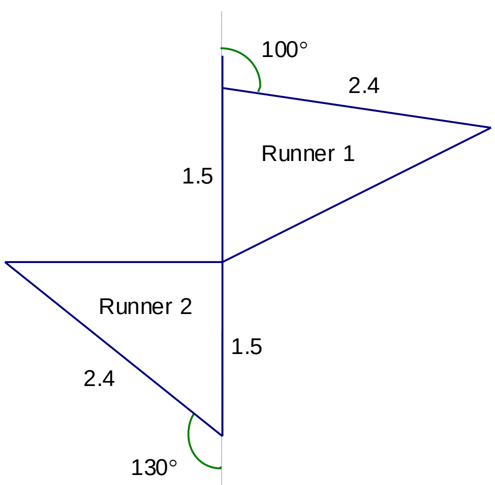

**a.** If $PR = PS$ and $m\angle QPR > m\angle QPS$, which is longer: $\overline{SQ}$ or $\overline{RQ}$?

$\overline{RQ}$ by the SAS Inequality (Hinge) Theorem

**b.** If $PR = PS$ and $RQ < SQ$, which is larger: $\angle RPQ$ or $\angle SPQ$?

$\angle SPQ$ by the SSS Inequality (Hinge Converse) Theorem

**Ex 3:** Two runners start together and run in opposite directions. Each one goes 1.5 miles, changes direction, and goes 2.4 miles. The first runner starts due north and runs. $100^\circ$ towards the east. The other runner starts due south and turns $130^\circ$ towards the west. Both runners return to the starting point. Which runner ran farther? *Explain.*




Each triangle has side lengths 1.5 mi and 2.4 mi, and the angles between those sides are $80^\circ$ and $50^\circ$. The *Hinge Theorem*, the third side of the triangle for Runner 1 is longer, so Runner 1 ran further.

**Key Vocab:**

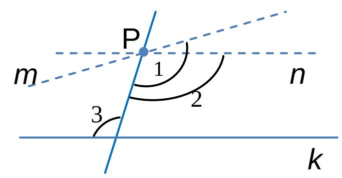
| | |
|---|---|
| **Indirect Proof** | A proof in which you prove that a statement is true by first assuming that its opposite is true. If this assumption leads to an impossibility or contradiction, then you have proved that the original statement is true. |

**Key Concept:**

### How to Write an Indirect Proof:

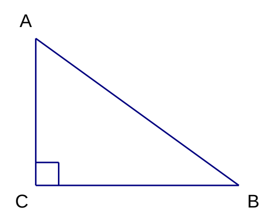
1. Identify the statement you want to prove. Assume temporarily that this statement is false by assuming that its opposite is true.
2. Reason logically until you reach a contradiction
3. Point out that the desired conclusion must be true because the contradiction proves the temporary assumption false.

**Ex 4:** Write an indirect proof of the Parallel Postulate:
*Through a point not on a line, there is exactly one line parallel to a given line.*




- Temporarily assume that *m* and *n* are both parallel to *k* through point *P*.
- By the alternate interior angles theorem, $\angle 3 \cong \angle 1$ and $\angle 3 \cong \angle 2$.
- Then, by the transitive property, $\angle 1 \cong \angle 2$.
- $\rightarrow\leftarrow$ But this statement contradicts that fact that $m\angle 1 > m\angle 2$.
- Therefore, the temporary assumption must be false. Only one line through point *P* may be parallel to *k*

**Ex 5:** Write an indirect proof of the Corollary of the Triangle Sum Theorem.
*The acute angles of a right triangle are complementary.*




- Temporarily assume that $m\angle 1 + m\angle 2 \neq 90^\circ$. This presents two cases:
  - **Case 1:** $m\angle 1 + m\angle 2 > 90^\circ$
    - If $m\angle 1 + m\angle 2 > 90^\circ$, then $m\angle C < 90^\circ$ because of the *Triangle Sum Theorem.* $\rightarrow\leftarrow$ This contradicts the fact that $\angle C$ is right angle.
  - **Case 2:** $m\angle 1 + m\angle 2 < 90^\circ$
    - If $m\angle 1 + m\angle 2 < 90^\circ$, then $m\angle C > 90^\circ$ because of the *Triangle Sum Theorem.* $\rightarrow\leftarrow$ This contradicts the fact that $\angle C$ is a right angle.
- Therefore, the assumption must be false. $m\angle 1 + m\angle 2 = 90^\circ$

**Closure:**

- Describe the difference between the Hinge Theorem and its Converse.

The Hinge Theorem is also called the SAS Inequality Theorem and it makes a conclusion about the third side of a triangle.

The Converse of the Hinge Theorem is also called the SSS Inequality Theorem and it makes a conclusion about the included angle of a triangle.